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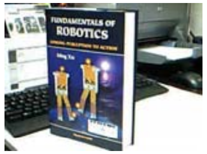

MP4006 Robotics

Lecture 3-5

(Robot Motion Planning)

Trajectory Generating

presented by

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Outline

- Revision: Path Generating
- Concept of Trajectory
- Equations of Trajectory
- Trapezoidal Velocity Profile
- Case Studies

Revision: Path Generating

Fact

- A robot can self-generate a smooth path if it knows:
  - Initial pose
  - Final pose
  - Work-space

Example

Vision-guided Parking

Fact

- A robot can perform a task in:
  - Real environment
  - Real-time

Discussion

- How to add time constraints to a path so that a robot can follow it continuously?

Concept of Trajectory

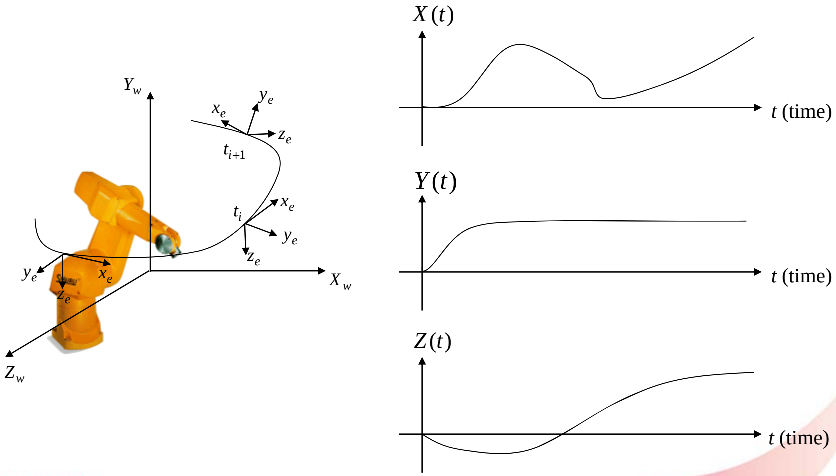
Definition

- By definition, a series of positions with time constraints (e.g. time instances, velocities, or accelerations) is called a trajectory.

Representation

- A trajectory is usually represented by the time functions of coordinates (e.g. coordinates of positions, coordinate of angle).

Example



Discussion

- How to transform a path into a trajectory?

Equations of Trajectory

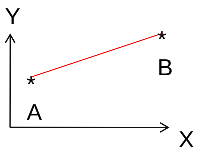
Fact

- All robotics applications are under time constraints in one way, or another.

Start

Question

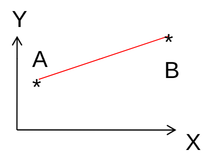
- If a robot's hand moves from A to B by following a straight line, what should be the time function which describes the trajectory of the robot's hand?



Answer

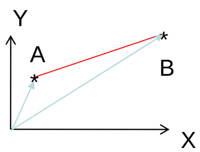
Path

$$y = ax + b$$



Trajectory

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} + \frac{\vec{r}_B - \vec{r}_A}{\|AB\|} d(t)$$

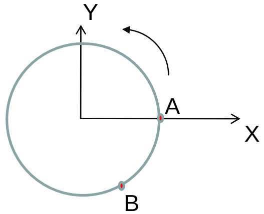


Travelled Distance

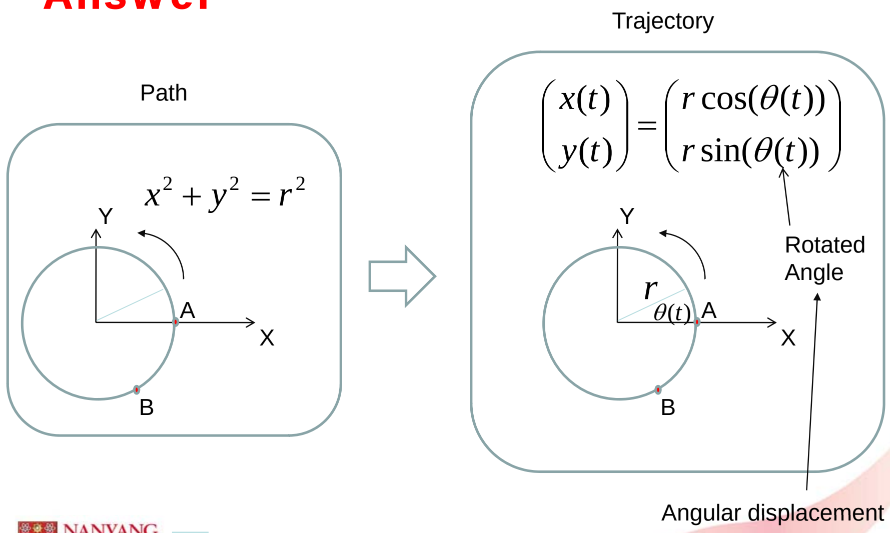
Linear displacement

Question

- If a robot's hand moves from A to B by following a circular curve, what should be the time function which describes the trajectory of the robot's hand?



Answer



Definition

- The time functions of coordinates where a robot's hand (or base) will pass through are called Equations of trajectory.

Discussion

- How to determine the time function of travelled distance or rotated angle?

Trapezoidal Velocity Profile

Question

- What is a one-dimensional motion?

Answer

- It is a motion related to the change of a single coordinate.

Question

- Could we consider a travelled distance or rotated angle to be a generalized coordinate?

Answer

- Yes, we can name it as a generalized coordinate denoted by g(t).

Question

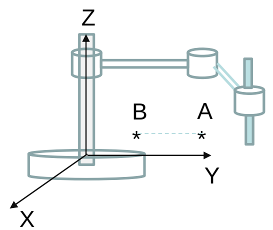
- What is the simplest generic motion of g(t) in one-dimension?

Answer

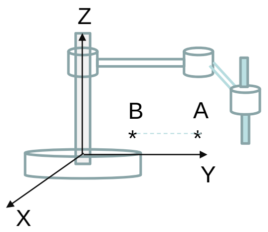
- It is the motion with constant acceleration!

Example

- A robot moves its hand from A to B by following a straight line. During the motion, the hand undergoes a motion with a constant acceleration. What is the equation describing the travelled distance by the robot's hand?



Answer



Speed at time t is :

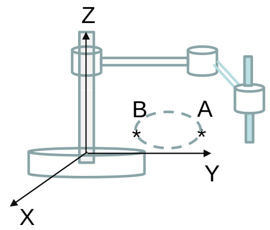
$$\dot{g}(t) = \dot{g}(t_0) + a_g t$$

Travelled distance at time t is :

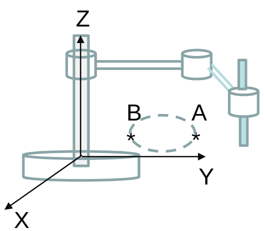
$$g(t) = \dot{g}(t_0)t + \frac{1}{2}a_g t^2$$

Example

- A robot moves its hand from A to B by following a circular curve. During the motion, the hand undergoes a motion with a constant angular acceleration. What is the equation describing the rotated angle about the center of the circular curve by the robot's hand?



Answer



Speed at time t is :

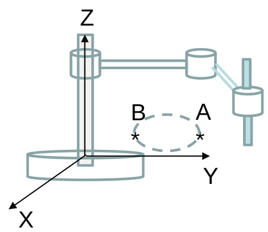
$$\dot{g}(t) = \dot{g}(t_0) + a_g t$$

Rotated angle at time t is :

$$g(t) = \dot{g}(t_0)t + \frac{1}{2}a_g t^2$$

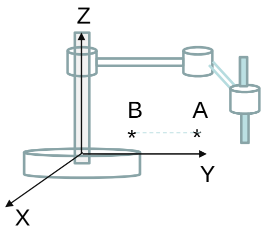
Discussion

- In the previous two examples, will the robot's hand stop at point B?



Question

- How to make a robot's hand to stop at point B?

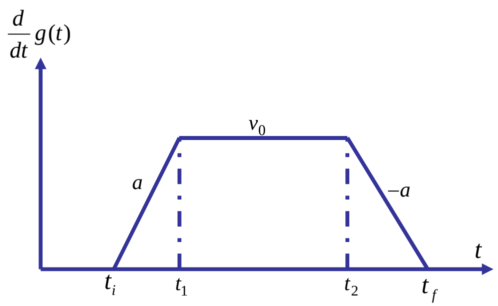


Answer

- We must design a motion profile consisting of three consecutive motions of:
  - 1. Constant acceleration
  - 2. Constant speed
  - 3. Constant deceleration

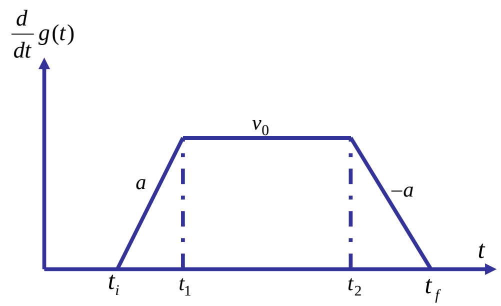
Definition of Trapezoidal Velocity Profile

- A speed function which consists of constant acceleration, constant speed and constant deceleration is called a Trapezoidal Velocity Profile.



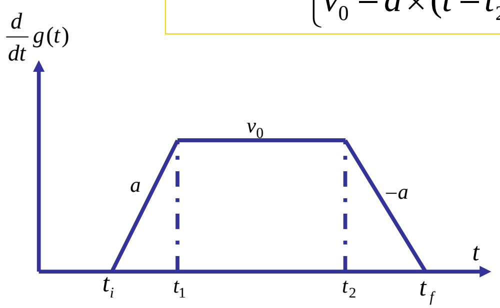
Question

- What are the equations which describe a trapezoidal velocity profile?



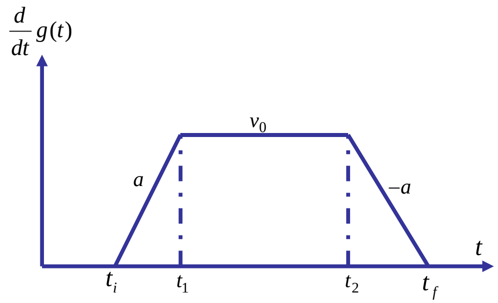
Answer

$$\frac{d}{dt}g(t)=\begin{cases} a\times(t-t_i) & \text{if } t_i \le t \le t_1 \\ v_0 & \text{if } t_1 \le t \le t_2 \\ v_0 - a\times(t-t_2) & \text{if } t_2 \le t \le t_f \end{cases}$$



Question

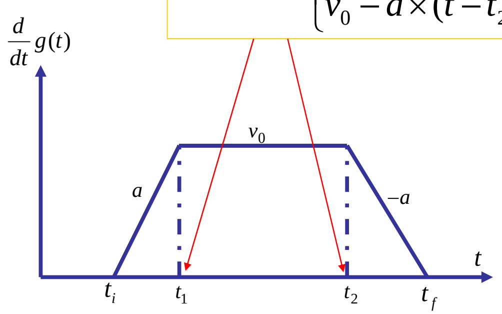
- How many constraints or conditions are there for a trapezoidal velocity profile to meet?



Answer

- Three!

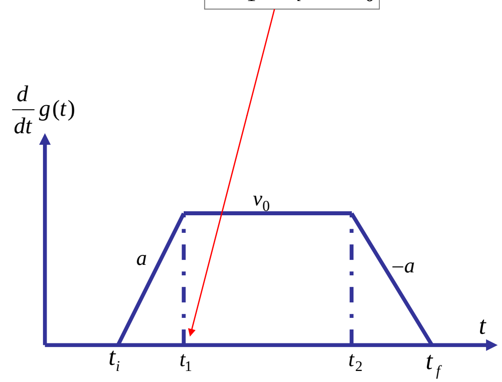
$$\frac{d}{dt}g(t)=\begin{cases} a\times(t-t_i) & \text{if } t_i \le t \le t_1 \\ v_0 & \text{if } t_1 \le t \le t_2 \\ v_0 - a\times(t-t_2) & \text{if } t_2 \le t \le t_f \end{cases}$$



Constraint 1: End of acceleration

Equation 1

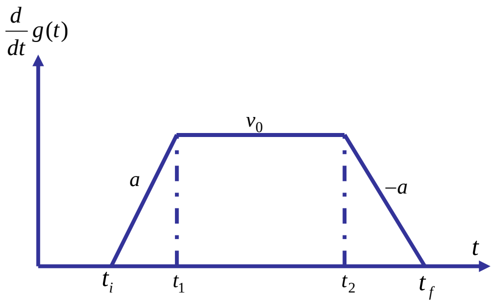
$$a(t_1 - t_i) = v_0$$



Constraint 2: Start of deceleration

Equation 2

$$v_0 - a(t_f - t_2) = 0$$



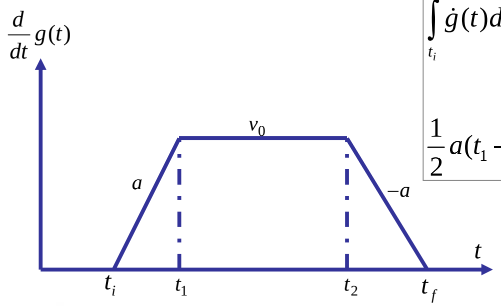
Constraint 3: Total displacement

Equation 3

$$\int_{t_i}^{t_f} \dot{g}(t)dt = D$$ ← Travelled Distance or Rotated Angle

$$\int_{t_i}^{t_1} \dot{g}(t)dt + \int_{t_1}^{t_2} \dot{g}(t)dt + \int_{t_2}^{t_f} \dot{g}(t)dt = D$$

$$\frac{1}{2}a(t_1 - t_i)^2 + v_0(t_2 - t_1) + \frac{1}{2}a(t_2 - t_f)^2 = D$$



Question

- What is the usage of the three constraints of a trapezoidal velocity profile?

Answer

- Three constraints allow to solve for three unknown parameters!

Example 1: Desired cruise velocity

$$a(t_1 - t_i) = v_0$$

$$v_0 - a(t_f - t_2) = 0$$

$$\frac{1}{2}a(t_1 - t_i)^2 + v_0(t_2 - t_1) + \frac{1}{2}a(t_2 - t_f)^2 = D$$

Three equations

⇩

Solve for three unknowns (a, t_1, t_2)

Example 2: Shortest cycle time

$$a(t_1 - t_i) = v_0$$

$$v_0 - a(t_f - t_2) = 0$$

$$\frac{1}{2}a(t_1 - t_i)^2 + v_0(t_2 - t_1) + \frac{1}{2}a(t_2 - t_f)^2 = D$$

Three equations

⇩

Solve for three unknowns (v_0, t_1, t_2)

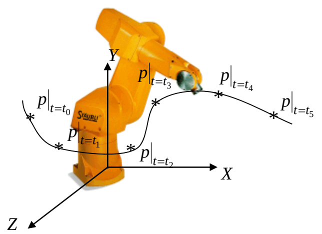
Discussion

- Could a robot plan trajectories of complex motions?

Case Studies

Case 1: Trajectory of polylines

What should be the equations of trajectory, if a robot moves the tool-tip to go through a series of locations one by one?



What should be the trajectory? ⟶ $p(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

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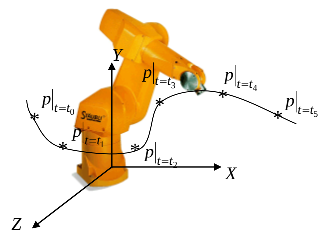
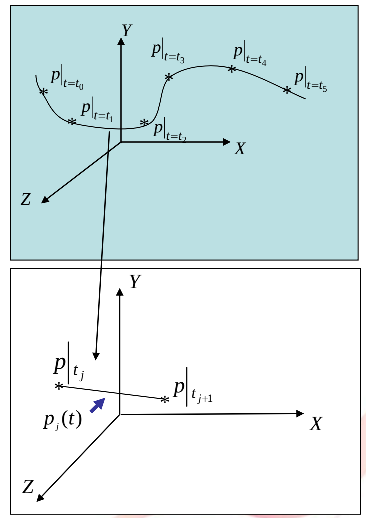
Answer

- We divide the whole trajectory into a series of segments.





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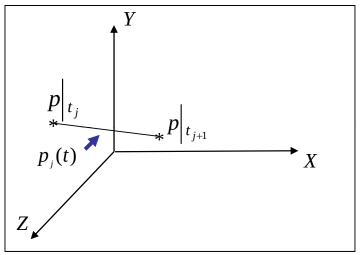
Answer (continued)

- We design an equation of trajectory for each segment.



Equation of trajectory j:

$$p_j(t)=\begin{cases} p|_{t_j} + \dfrac{p|_{t_{j+1}} - p|_{t_j}}{\left\| p|_{t_{j+1}} - p|_{t_j} \right\|} \bullet g(t), & \text{if } t_j \le t \le t_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

where $g(t)$ is the traveled distance within $[t_j, t]$.

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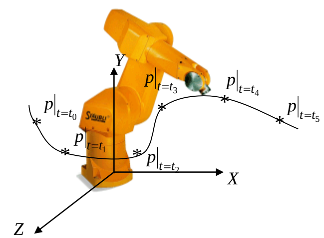
Answer (continued)

- Finally, we can combine trajectories 0, 1, 2, … together.

Given a sequence of N waypoints, the trajectory will be:

$$p(t)=\sum_{j=0}^{N-1} p_j(t)$$



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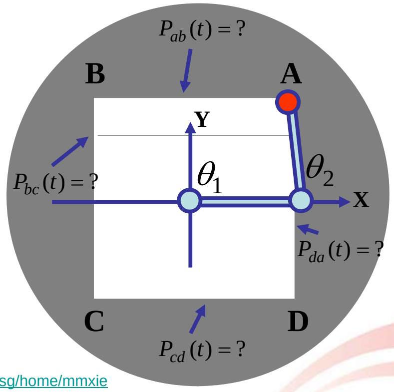
Example

- We use a robot arm with two links to carry a laser gun in order to a cut a square. When the laser gun is turned on, the robot must move the laser gun at a desired velocity. What should be the equations, which describe the trajectories of the laser gun?



Answer

- Input:

$$P_a(t_a)=\begin{pmatrix}x_a\\y_a\end{pmatrix} \rightarrow P_b(t_b)=\begin{pmatrix}x_b\\y_b\end{pmatrix}$$

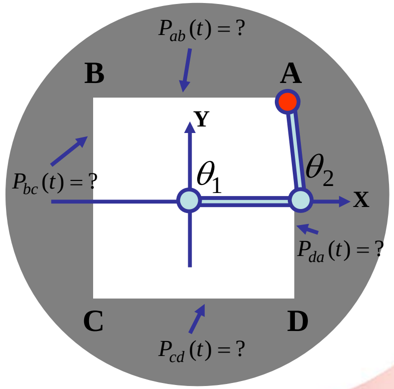
$$\downarrow$$

$$P_d(t_d)=\begin{pmatrix}x_d\\y_d\end{pmatrix} \leftarrow P_c(t_c)=\begin{pmatrix}x_c\\y_c\end{pmatrix}$$

$$\downarrow$$

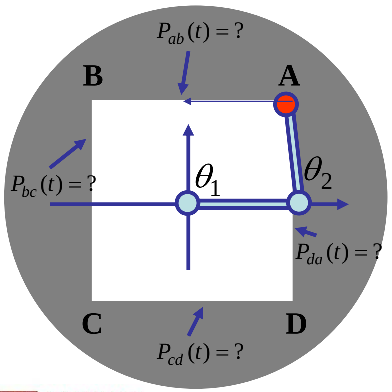
$$P_a(t_e)=\begin{pmatrix}x_a\\y_a\end{pmatrix}$$

Desired velocity: v_0



Answer (continued)

- Equation of Trajectory from A to B:



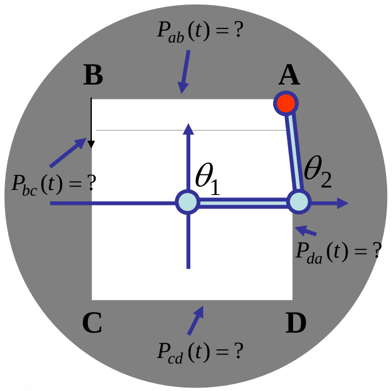
$$P_{ab}(t)=P_a+\frac{P_b-P_a}{\|P_b-P_a\|}g(t)$$

with the travelled distance $g(t)$ to be:

$$g(t)=\begin{cases}0, & \text{if } t=t_a\\ \|P_b-P_a\|, & \text{if } t=t_b\end{cases}$$

Answer (continued)

- Equation of Trajectory from B to C:



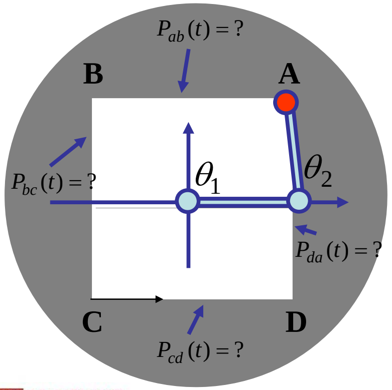
$$P_{bc}(t)=P_b+\frac{P_c-P_b}{\|P_c-P_b\|}g(t)$$

with the travelled distance $g(t)$ to be:

$$g(t)=\begin{cases}0, & \text{if } t=t_b\\ \|P_c-P_b\|, & \text{if } t=t_c\end{cases}$$

Answer (continued)

- Equation of Trajectory from C to D:



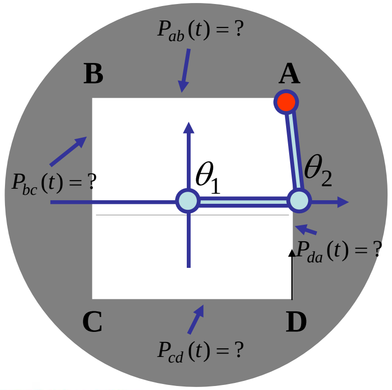
$$P_{cd}(t) = P_c + \frac{P_d - P_c}{\|P_d - P_c\|} g(t)$$

with the travelled distance $g(t)$ to be:

$$g(t) = \begin{cases} 0, & \text{if } t = t_c \\ \|P_d - P_c\|, & \text{if } t = t_d \end{cases}$$

Answer (continued)

- Equation of Trajectory from D to A:



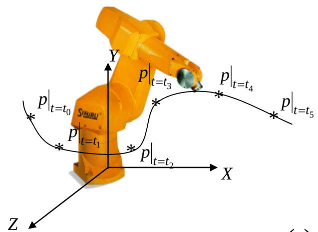
$$P_{da}(t) = P_d + \frac{P_a - P_d}{\|P_a - P_d\|} g(t)$$

with the travelled distance $g(t)$ to be:

$$g(t) = \begin{cases} 0, & \text{if } t = t_d \\ \|P_a - P_d\|, & \text{if } t = t_a \end{cases}$$

Case 2: Trajectory of curves

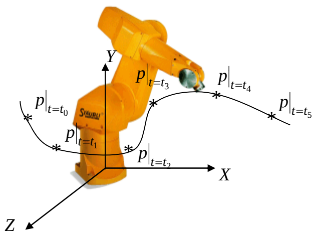
What will be the equations of trajectory, if a robot moves the tool-tip to go through a series of locations continuously ?



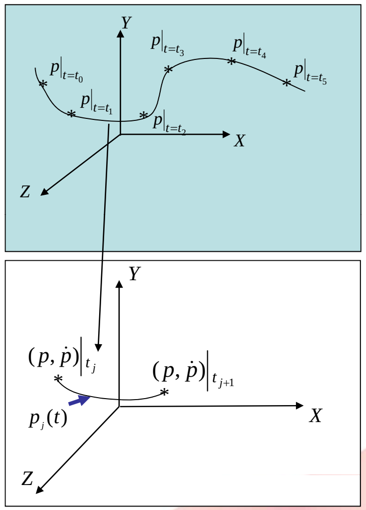
$$p_j(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Answer

- We divide the whole trajectory into a series of segments.

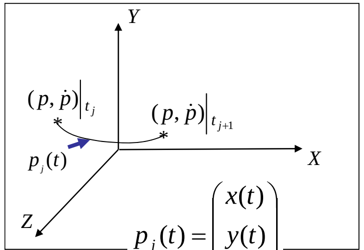


$$p_j(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$



Answer (continued)

- We choose an equation of trajectory for each segment.



$$p_j(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Curve of 3rd Order Polynormials (eg., Cubic Polynormial):

$$p_j(t) = \begin{cases} a_3 t^3 + a_2 t^2 + a_1 t + a_0, & \text{if } t_j \le t \le t_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

with

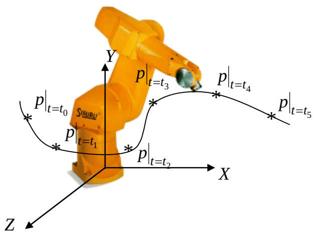
$$a_0 = \begin{pmatrix} a_{01} \\ a_{02} \\ a_{03} \end{pmatrix}, \quad a_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}$$

Answer (continued)

- Finally, we combine the trajectories 0, 1, 2, …, together.

Given a sequence of N pass-points, the trajectory will be:

$$p(t) = \sum_{j=0}^{N-1} p_j(t)$$



Question

- How to determine the coefficients inside an equation of trajectory?

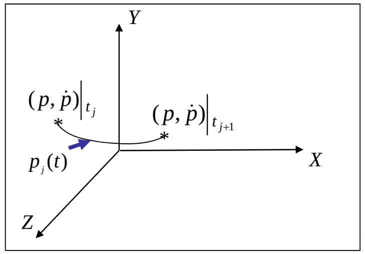
Curve of 3rd Order Polynormials (eg., Cubic Polynormial):

$$p_j(t) = \begin{cases} a_3 t^3 + a_2 t^2 + a_1 t + a_0, & \text{if } t_j \le t \le t_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

with

$$a_0 = \begin{pmatrix} a_{01} \\ a_{02} \\ a_{03} \end{pmatrix}, \quad a_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}$$

Hint 1:



Within the time interval $[t_j, t_{j+1}]$:

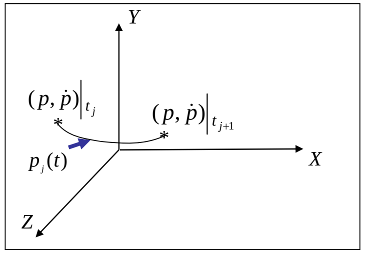
$$p_j(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

$$\dot{p}_j(t) = \frac{dp_j(t)}{dt} = 3a_3 t^2 + 2a_2 t + a_1$$

Continuity conditions at waypoint t j:

$$\begin{cases} p_j(t_j) = p|_{t=t_j} = a_3 t_j^3 + a_2 t_j^2 + a_1 t_j + a_0 \\ \dot{p}_j(t_j) = \dot{p}|_{t=t_j} = 3a_3 t_j^2 + 2a_2 t_j + a_1 \end{cases}$$

Hint 2:



Within the time interval $[t_j, t_{j+1}]$:

$$p_j(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

$$\dot p_j(t) = \frac{dp_j(t)}{dt} = 3a_3t^2 + 2a_2t + a_1$$

Continuity conditions at waypoint j+1:

$$\begin{cases} p_j(t_{j+1}) = p\big|_{t=t_{j+1}} = a_3t_{j+1}^3 + a_2t_{j+1}^2 + a_1t_{j+1} + a_0 \\ \dot p_j(t_{j+1}) = \dot p\big|_{t=t_{j+1}} = 3a_3t_{j+1}^2 + 2a_2t_{j+1} + a_0 \end{cases}$$

Hint 3: System of equations

Continuity conditions at waypoint t j:

$$\begin{cases} p_j(t_j) = p\big|_{t=t_j} = a_3t_j^3 + a_2t_j^2 + a_1t_j + a_0 \\ \dot p_j(t_j) = \dot p\big|_{t=t_j} = 3a_3t_j^2 + 2a_2t_j + a_1 \end{cases}$$

Continuity conditions at waypoint j+1:

$$\begin{cases} p_j(t_{j+1}) = p\big|_{t=t_{j+1}} = a_3t_{j+1}^3 + a_2t_{j+1}^2 + a_1t_{j+1} + a_0 \\ \dot p_j(t_{j+1}) = \dot p\big|_{t=t_{j+1}} = 3a_3t_{j+1}^2 + 2a_2t_{j+1} + a_0 \end{cases}$$

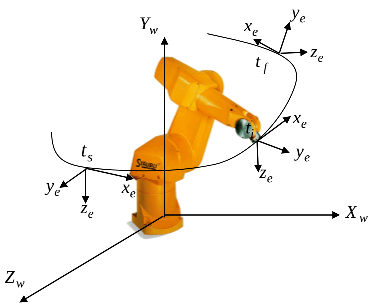
4 sets of equations

⇓

4 vectors of coefficients

Case 3: Trajectory of orientation

- A coordinate system is assigned to the tool attached to a robot's hand. When the tool moves from time ts to time tf, what will be the equation which describes the trajectory of the orientation of the tool?

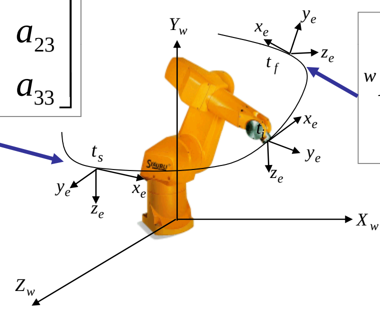


Answer

- First to determine the tool's orientations at times ts and tf, with respect to a common world coordinate system.

$${}^wR_{t_s} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Ts -> World



$${}^wR_{t_f} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Tf -> World

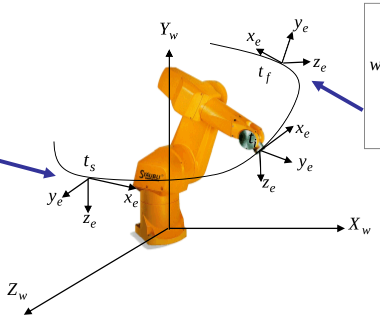
(to continue …)

Answer (continued)

- Step 2: To determine the tool's orientation at time tf, with respect to the tool's orientation at time ts.

$$ {}^{w}R_{t_s} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} $$

Ts -> World



$$ {}^{w}R_{t_f} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} $$

Tf -> World

Tf -> Ts = (Tf -> World) followed by (World -> Ts)

(to continue …)

Tf -> Ts = (Tf -> World) followed by (World -> Ts)

$$ {}^{t_s}R_{t_f} = \{{}^{t_s}R_w\} \bullet \{{}^{w}R_{t_f}\} = \{{}^{w}R_{t_s}\}^{-1} \bullet \{{}^{w}R_{t_f}\} $$

$$ = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} $$

$$ {}^{t_s}R_{t_f} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} $$

Answer (continued)

- Step 3: To determine the equivalent axis of rotation and equivalent angle of rotation.

$$ R = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} $$

Equivalent Axis of Rotation:

$$ \vec{r}\big|_{t_s t_f} = \begin{bmatrix} c_{32} - c_{23} \\ c_{13} - c_{31} \\ c_{21} - c_{12} \end{bmatrix} $$

Equivalent Angle of Rotation:

$$ \Delta\theta = \arccos\left(\frac{c_{11} + c_{22} + c_{33} - 1}{2}\right) $$

Answer (continued)

- Step 4: To design the equation of trajectory of the rotated angle.

Equation of trajectory for rotation angle:

$$ \theta(\mathrm{t}) = \theta\big|_{t=t_s} + \frac{\theta\big|_{t=t_f} - \theta\big|_{t=t_s}}{\left\| \theta\big|_{t=t_f} - \theta\big|_{t=t_s} \right\|} g(t) $$

with g(t) being the rotated angle at time t.

Summary

- Revision: Path Generating
- Concept of Trajectory
- Equations of Trajectory
- Trapezoidal Velocity Profile
- Case Studies

Enjoy Your Study of Robotics